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Traveling-Wave-Type Gravitational Soliton Solutions

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With the traveling-wave condition of Yan and Ge, the traveling-wave-type gravitational two-soliton solutions are generated from a flat metric by using the inverse scattering method (ISM) of Belinsky and Zakharov (BZ). It is shown that when the traveling-wave condition is added to the condition required by the BZ technique the exact general solutions of the vacuum gravitational field equations can be found by straightforward integration; in the general solutions there are three arbitrary functions. A special solution with solitonic character in the ordinary sense is also given.

1. INTRODUCTION

In view of the fact that it is very difficult to solve the gravitational field equations, a technique for generating a new solution from a known one is quite important. Among the techniques for the generation of solutions developed in recent years, the inverse scattering method (ISM) of Belinsky and Zakharov (BZ) (1978) is one of the most effective. As long as the space-time manifold admits a pair of commuting Killing vectors, the BZ technique can be used to generate exact solutions. A large number of interesting solutions have been found and investigated by use of the BZ technique (Belinsky and Zakharov, 1978, 1980; Belinsky and Ruffini, 1980; Car and Verdaguer, 1983; Ibanez and Verdaguer, 1985, 1986; Bruckman, 1986).

Yan and Ge (1987) added a traveling-wave condition to the condition required by the BZ technique in order to obtain soliton solutions which possess traveling-wave character, and constructed a class of traveling-wave soliton solutions from the general form of the Bondi plane-wave metric which serves as a "seed" by using the BZ technique.

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In the present paper we first construct traveling-wave-type two-soliton solutions from a still simpler (flat) "seed" metric by using the BZ technique and examine the corresponding curvature; then we point out emphatically that under the condition considered by Yan and Ge (1987) the general solutions of the vacuum field equations can be obtained by the use of quadrature and construct a special solution for which the corresponding curvature possesses solitonic behavior.

2. THE BZ TECHNIQUE

Let us first briefly describe the BZ technique. If the space-time manifold admits two commuting spacelike Killing vectors, the line element can be written as

$$ds^2 = f(z, t)(-dt^2 + dz^2) + \gamma_{ab}(z, t)\, dx^a\, dx^b, \qquad a, b = 1, 2 \tag{2.1}$$

Putting

$$\xi = \frac{1}{2}(z + t), \qquad \eta = \frac{1}{2}(z - t) \tag{2.2}$$

and defining

$$\alpha^2 = \mathrm{Det}(\gamma), \qquad \mathcal{A} = -\alpha\gamma_{,\xi}\gamma^{-1}, \qquad \mathcal{B} = \alpha\gamma_{,\eta}\gamma^{-1} \tag{2.3}$$

where γ is a 2×2 matrix with γ_{ab} as its elements and γ^{-1} is the inverse of γ, we can write the vacuum Einstein equations as follows:

$$\mathcal{A}_{,\eta} + \mathcal{B}_{,\xi} = 0 \tag{2.4a}$$

$$(\ln f)_{,\xi} = (\ln \alpha)_{,\xi\xi}/(\ln \alpha)_{,\xi} + \mathrm{Tr}(\mathcal{A}^2)/4\alpha\alpha_{,\xi} \tag{2.4b}$$

$$(\ln f)_{,\eta} = (\ln \alpha)_{,\eta\eta}/(\ln \alpha)_{,\eta} + \mathrm{Tr}(\mathcal{B}^2)/4\alpha\alpha_{,\eta} \tag{2.4c}$$

In the BZ technique, equation (2.4a) is associated with a linear eigenvalue problem:

$$\boldsymbol{\Psi}_{,\xi} - \frac{2\lambda\alpha_{,\xi}}{\lambda - \alpha}\boldsymbol{\Psi}_{,\lambda} = \frac{\mathcal{A}}{\lambda - \alpha}\boldsymbol{\Psi} \tag{2.5a}$$

$$\boldsymbol{\Psi}_{,\eta} + \frac{2\lambda\alpha_{,\eta}}{\lambda + \alpha}\boldsymbol{\Psi}_{,\lambda} = \frac{\mathcal{B}}{\lambda + \alpha}\boldsymbol{\Psi} \tag{2.5b}$$

where λ is a complex spectral parameter and $\boldsymbol{\Psi}$ a complex 2×2 matrix which satisfies

$$\gamma(\xi, \eta) = \boldsymbol{\Psi}(\xi, \eta; \lambda = 0), \qquad \boldsymbol{\Psi}(\bar{\lambda}) = \boldsymbol{\Psi}(\lambda) \tag{2.6}$$

where a bar on a letter represents the complex conjugate. Introducing the scattering matrix by

$$\Psi = \chi(\xi, \eta; \lambda)\Psi_0, \qquad \bar{\chi}(\bar{\lambda}) = \chi(\lambda), \qquad \lim_{\lambda \to \infty} \chi(\lambda) = \mathcal{I} \tag{2.7}$$

with Ψ_0 as the solution of equations (2.5) which corresponds a "seed" metric γ_0 in the sense of (2.6), and carrying out the spectral decomposation in the complex λ plane, one has

$$\chi(\lambda) = \mathcal{I} + \sum_{k=1}^{n} \frac{\mathcal{R}_k}{\lambda - \mu_k} \tag{2.8}$$

where $\mathcal{I}$ is the unitary matrix, and the elements of $\mathcal{R}_k$ are defined by

$$(\mathcal{R}_k)_{ab} = \sum_{l=1}^{n} \frac{(\Gamma^{-1})_{lk} M_c^{(k)} (\gamma_0)_{ca} M_b^{(l)}}{\mu_l} \tag{2.9}$$

$$M_c^{(k)} = (M_0)_d^{(k)} (\Psi_0^{-1}(\xi, \eta; \mu_k))_{dc} \tag{2.10}$$

$$\Gamma_{kl} = \frac{M_c^{(k)} (\gamma_0)_{cb} M_b^{(l)}}{\mu_k \mu_l - \alpha^2} \tag{2.11}$$

$$\mu_k = \omega_k - \beta + [(\omega_k - \beta)^2 - \alpha^2]^{1/2}, \qquad k = 1, 2, \ldots \tag{2.12}$$

and $M_c^{(k)}$, ω_k are arbitrary constants; β is the harmonic conjugate of α, which is defined by

$$\alpha = a(\xi) + b(\eta), \qquad \beta = a(\xi) - b(\eta) \tag{2.13}$$

(Here, note that $\alpha_{,\xi\eta} = 0$.) Finally, the new solution is given by

$$\gamma_{ab} = \alpha^{-n} \left(\prod_{k=1}^{n} \mu_k \right) \left[(\gamma_0)_{ab} - \sum_{k,l=1}^{n} \frac{(\Gamma^{-1})_{kl} M_c^{(k)} (\gamma_0)_{ca} M_d^{(k)} (\gamma_0)_{db}}{\mu_k \mu_l} \right] \tag{2.14a}$$

$$f = c f_0 \alpha^{-n^2/2} \frac{\left(\prod_{k=1}^{n} \mu_k \right)^{n+1}}{\prod_{k,l=1, k>l}^{n} (\mu_k - \mu_l)^2} \operatorname{Det}(\Gamma) \tag{2.14b}$$

which is the so-called n-soliton solution.

3. THE TWO-SOLITON TRAVELING WAVE ON THE FLAT BACKGROUND

In order to obtain traveling wave solutions, Yan and Ge (1987) introduced the following traveling-wave condition:

$$g_{\mu\nu}(z, t) = g_{\mu\nu}(u), \qquad u = z - t = 2\eta \tag{3.1}$$

Hereafter we restrict ourselves to the traveling-wave case. We take the line element of the flat space-time

$$ds^2 = -dt^2 + dz^2 + u^2(dx^2 + dy^2) \tag{3.2}$$

as a "seed" metric. Thus, from equations (2.3) we have $f_0 = 1$, $\gamma_0 = u^2\mathcal{I}$, $\alpha = u^2$, $\mathcal{A}_0 = 0$, and $\mathcal{B}_0 = 4u\mathcal{I}$. Substituting these results into (2.5) and solving, we obtain

$$\mathbf{\Psi}_0(u; \lambda) = (u^2 - \lambda)\mathcal{I} \tag{3.3a}$$

From equation (2.12) it is easily seen that

$$\mu_k = \omega_k + u^2 + (\omega_k^2 + 2\omega_k u^2)^{1/2} \tag{3.4a}$$

and

$$u^2 - \mu_k = (2\omega_k\mu_k)^{1/2} \tag{3.4b}$$

The combination of (3.4a) with (3.4b) gives

$$\mathbf{\Psi}_0(u; \mu_k) = (2\omega_k\mu_k)^{1/2}\mathcal{I} \tag{3.3b}$$

When we consider the real pole trajectories and the case where $n = 2$, we have

$$m_1^{(1)} = \frac{c_1^{(1)}}{\sqrt{\mu_1}}, \qquad m_1^{(2)} = \frac{c_1^{(2)}}{\sqrt{\mu_2}}, \qquad m_2^{(1)} = \frac{c_2^{(1)}}{\sqrt{\mu_1}}, \qquad m_2^{(2)} = \frac{c_2^{(2)}}{\sqrt{\mu_2}} \tag{3.5}$$

$$\Gamma_{11} = \frac{[(c_1^{(1)})^2 + (c_2^{(1)})^2]u^2}{\mu_1(\mu_1^2 - u^4)} \tag{3.6a}$$

$$\Gamma_{12} = \Gamma_{21} = \frac{(c_1^{(1)}c_1^{(2)} + c_2^{(1)}c_1^{(2)})u^2}{(\mu_1\mu_2)^{1/2}(\mu_1\mu_2 - u^4)} \tag{3.6b}$$

$$\Gamma_{22} = \frac{[(c_1^{(2)})^2 + (c_2^{(2)})^2]u^2}{\mu_2(\mu_2^2 - u^4)} \tag{3.6c}$$

where $c_a^{(k)} = (m_0)_a^{(k)}/(2\omega_k)^{1/2}$.

Now we consider separately the two special cases for simplicity.

Case 1. $c_1^{(k)} = c_2^{(k)} = c_k$. In this case we have

$$m_1^{(1)} = m_2^{(1)} = \frac{c_1}{\sqrt{\mu_1}}, \qquad m_1^{(2)} = m_2^{(2)} = \frac{c_2}{\sqrt{\mu_2}}$$

$$\Gamma_{11} = \frac{2c_1^2u^2}{\mu_1(\mu_1^2 - u^4)}, \quad \Gamma_{12} = \Gamma_{21} = \frac{2c_1c_2u^2}{(\mu_1\mu_2)^{1/2}(\mu_1\mu_2 - u^4)}, \quad \Gamma_{22} = \frac{2c_2^2u^2}{\mu_2(\mu_2^2 - u^4)}$$

The new solutions are

$$\gamma_{11} = \gamma_{22} = \frac{1}{2}\left(\frac{\mu_1\mu_2}{u^2} + \frac{u^6}{\mu_1\mu_2}\right) \tag{3.7a}$$

$$\gamma_{12} = \gamma_{21} = \frac{1}{2}\left(\frac{\mu_1\mu_2}{u^2} - \frac{u^6}{\mu_1\mu_2}\right) \tag{3.7b}$$

$$f = \frac{4(\sqrt{\omega_1} + \sqrt{\omega_2})^2\mu_1\mu_2 u^4}{(\mu_1\mu_2 - u^4)^2(\omega_1 + 2u^2)^{1/2}(\omega_2 + 2u^2)^{1/2}} \tag{3.7c}$$

where we have determined that

$$c = 4\,\frac{\sqrt{\omega_1\omega_2}(\sqrt{\omega_1} + \sqrt{\omega_2})^2}{(c_1c_2)^2} \tag{3.8}$$

by the requirement that $\lim_{u\to\infty} f = 1$.

Case 2. $c_1^{(k)} = 0$, $c_2^{(k)} = \bar{c}_k$. Now, we have that

$$m_1^{(1)} = m_1^{(2)} = 0, \qquad m_2^{(1)} = \frac{\bar{c}_1}{\sqrt{\mu_1}}, \qquad m_2^{(2)} = \frac{\bar{c}_2}{\sqrt{\mu_2}}$$

$$\Gamma_{11} = \frac{\bar{c}_1^2 u^2}{\mu_1(\mu_1^2 - u^4)}, \qquad \Gamma_{12} = \Gamma_{21} = \frac{\bar{c}_1\bar{c}_2 u^2}{(\mu_1\mu_2)^{1/2}(\mu_1\mu_2 - u^4)},$$

$$\Gamma_{22} = \frac{\bar{c}_1^2 u^2}{\mu_2(\mu_2^2 - u^4)}$$

The new solutions read as follows:

$$\gamma_{11} = \frac{\mu_1\mu_2}{u^2}, \qquad \gamma_{22} = \frac{u^6}{\mu_1\mu_2}, \qquad \gamma_{12} = \gamma_{21} = 0 \tag{3.9a}$$

$$f = 4\,\frac{(\sqrt{\omega_1} + \sqrt{\omega_2})^2\mu_1\mu_2 u^4}{(\mu_1\mu_2 - u^4)^2(\omega_1 + 2u^2)^{1/2}(\omega_2 + 2u^2)^{1/2}} \tag{3.9b}$$

Similarly, the constant c has been replaced by

$$c = 16\,\frac{\sqrt{\omega_1\omega_2}(\sqrt{\omega_1} + \sqrt{\omega_2})^2}{(\bar{c}_1\bar{c}_2)^2} \tag{3.10}$$

It is well known that it is the curvature tensor that really indicates the existence of the gravitational field. In the geodesic deviation equations

$$\frac{d^2\delta x^i}{dt^2} = R^i_{0j0}\delta x^j \tag{3.11}$$

R^i_{0j0} corresponds to the tidal force. It is such a kind of force that one desires to detect in the search for gravitational waves. Therefore, we give R^i_{0j0}. The nonzero components for case 1 are

$$
\begin{aligned}
{}^1R^2_{010} = {} & \frac{1}{u^2(\mu_1 + u^2)^3(\mu_2 + u^2)^3} [9u^{12} + (\mu_1 + \mu_2)u^{10} - (\mu_1^2 + \mu_2^2 + 9\mu_1\mu_2)u^8 \\
& + \mu_1\mu_2(\mu_1^2 + \mu_2^2 + 9\mu_1\mu_2)u^4 - 4\mu_1^2\mu_2^2(\mu_1 + \mu_2)u^2 - 9\mu_1^3\mu_2^3]
\end{aligned} \tag{3.12}
$$

and for case 2,

$$
\begin{aligned}
{}^2R^2_{020} = -{}^2R^1_{010} = {} & \frac{6}{u^2(\mu_1 + u^2)^3(\mu_2 + u^2)^3} [u^{12} - (\mu_1^2 + \mu_2^2 + \mu_1\mu_2)u^8 \\
& + \mu_1\mu_2(\mu_1^2 + \mu_2^2 + \mu_1\mu_2)u^4 - \mu_1^3\mu_2^3]
\end{aligned} \tag{3.13}
$$

It can be seen that both ${}^1R^2_{010}$ and ${}^2R^2_{020}$ go to infinity as u tends zero and vice versa. The two cases represent two distinguishable polarizations of the gravitational wave.

4. THE GENERAL SOLUTION AND A CLASS OF *n*-SOLITON SPECIAL SOLUTIONS

The BZ method is a very effective way of generating a new solution from an old one. The new solution is called the soliton solution only because the soliton technique is used to find it, while the new solution does not necessarily have the features of the classical soliton, as remarked by Ibanez and Verdaguer (1985). When we reexamine the field equations (2.4a)–(2.4c) with the traveling-wave condition (3.1), the discussion of Section 3 might appear somewhat pedantic. Under the condition (3.1), equations (2.4a) and (2.4b) become identities and equation (2.4c) becomes

$$
(\ln f)' = \frac{(\ln \alpha)''}{(\ln \alpha)'} + \frac{\mathrm{Tr}(\mathcal{B}^2)}{16\alpha\alpha'} \tag{4.1}
$$

where the prime denotes differentiation with respect to u. Straightforward integrating gives the general solution of (4.1):

$$
f = c\left(\sqrt{\gamma_{11}\gamma_{22} - (\gamma_{12})^2}\right) \exp\left(-\int \frac{\gamma_{11}'\gamma_{22}' - (\gamma_{12}')^2}{(\gamma_{11}\gamma_{22} - (\gamma_{12})^2)}\, du\right) \tag{4.2}
$$

where c is a constant and γ_{11}, γ_{22}, and γ_{12} are arbitrary functions.

Now we want to construct a kind of special solution with physically solitonic behavior. As mentioned above, the curvature tensor characterizes the gravitational field; its tetrad components are measurable. Hence, we choose γ_{11}, γ_{22}, and γ_{12} so that the tetrad components $R^{\hat{i}}_{\hat{0}\hat{j}\hat{0}}$ take the n-soliton form. For this purpose, we choose

$$\gamma_{12} = 0, \qquad \gamma_{11} = \alpha^{1+\sqrt{3}}, \qquad \gamma_{22} = \alpha^{1-\sqrt{3}} \tag{4.3}$$

where α is to be determined. Then we have

$$f = c\alpha\alpha' \tag{4.4}$$

The nonzero components of R^{i}_{0j0} are

$$R^{1}_{010} = -R^{2}_{020} = -\frac{\sqrt{3}}{2}\frac{\alpha'^2}{\alpha^2} \tag{4.5}$$

Introducing the tetrad

$$\begin{aligned} \omega^{\mu}_{\hat{0}} &= ((c\alpha\alpha')^{1/2}, 0, 0, 0), \qquad \omega^{\mu}_{\hat{1}} = (0, \alpha^{-\sqrt{3}/2}, 0, 0) \\ \omega^{\mu}_{\hat{2}} &= (0, 0, \alpha^{\sqrt{3}/2}, 0), \qquad \omega^{\mu}_{\hat{4}} = (0, 0, 0, (c\alpha\alpha')^{1/2}) \end{aligned} \tag{4.6}$$

we find the corresponding tetrad components

$$R^{\hat{1}}_{\hat{0}\hat{1}\hat{0}} = -\frac{\sqrt{3}\alpha'}{2c\alpha^3} \tag{4.7}$$

If we want $R^{\hat{1}}_{\hat{0}\hat{1}\hat{0}}$ to take the n-soliton form

$$R^{\hat{1}}_{\hat{0}\hat{1}\hat{0}} = -\frac{\sqrt{3}}{4c} A \sum_{k=1}^{n} \operatorname{sech}^2(u - k\delta) \tag{4.8}$$

we only need to put

$$\alpha = -\left[A \sum_{k=1}^{n} \operatorname{th}(u - k\delta) + \bar{c}\right]^{1/2} \tag{4.9}$$

To sum up, if the space-time manifold admits two commuting spacelike Killing vectors and satisfies the so-called traveling-wave condition, the exact general solution of the vacuum field equations can be found by straightforward integration; there are three free functions in it. A special solution with a form involving the n-soliton can be constructed by choosing suitably those free functions.

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